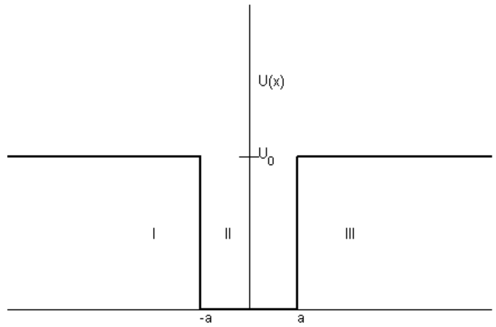
The Finite Square Well Potential
Treatment following "Introductory Quantum Mechanics" by Liboff.



We want to find solutions to the time independent Schrödinger equation

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \tag{1}$$

Using the potential in each of the three regions gives the wave functions

$$\psi_I = Ae^{\kappa x} \tag{2}$$
$$\psi_{II} = Be^{ikx} + Ce^{-ikx} \tag{3}$$
$$\psi_{III} = De^{-\kappa x} \tag{4}$$
$$\tag{5}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{6}$$
$$\kappa = \frac{\sqrt{2m(U_0 - E))}}{\hbar} \tag{7}$$
$$\tag{8}$$

and are related by

$$k^2 + \kappa^2 = \frac{2m|U_0|}{\hbar} \tag{9}$$

The first derivatives are

$$\psi'_I = \kappa Ae^{\kappa x} \tag{10}$$
$$\psi'_{II} = ikBe^{ikx} + -ikCe^{-ikx} \tag{11}$$
$$\psi'_{III} = -\kappa De^{-\kappa x} \tag{12}$$
$$\tag{13}$$

Both the wave functions and first derivatives must match at the boundary regions, leading to the following four equations:

$$\begin{aligned} Ae^{-\kappa x} &= Be^{-ika} + Ce^{ika} && (14)\\ Be^{ika} + Ce^{-ika} &= De^{-\kappa a} && (15)\\ \kappa Ae^{-\kappa x} &= ik(Be^{-ika} - Ce^{ika}) && (16)\\ ik(Be^{ika} - Ce^{-ika}) &= -\kappa De^{-\kappa x} && (17)\\ & && (18) \end{aligned}$$

This is as far as I would expect you to go for our physics 262 class, but here is a summary of the following analysis...
We can write the four above equations in matrix form:

$$\begin{pmatrix} e^{-\kappa x} & -e^{-ika} & -e^{ika} & 0 \\ 0 & e^{ika} & e^{-ika} & -e^{-\kappa a} \\ \kappa e^{-\kappa x} & -ike^{-ika} & ike^{ika} & 0 \\ 0 & ike^{ika} & -ike^{-ika} & \kappa e^{-\kappa x} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

By Cramer's Rule, there is only a non-trivial solution when the determinant of the 4×4 matrix is zero. With some algebra, it can be shown that this is only true when:

$$k\cot(ka) = -\kappa \tag{19}$$

or

$$k\tan(ka) = \kappa \tag{20}$$

Each case above restricts the possible values of the coefficients A, B, C, D leading to the odd wave functions

$$\left.\begin{aligned} \psi_I &= -2iB\sin(ka)e^{\kappa(x+a)} \\ \psi_{II} &= 2iB\sin(kx) \\ \psi_{III} &= 2iB\sin(ka)e^{-\kappa(x-a)} \end{aligned}\right\} k\cot(ka) = -\kappa$$

and the even wave functions

$$\left.\begin{aligned} \psi_I &= 2B\cos(ka)e^{\kappa(x+a)} \\ \psi_{II} &= 2B\cos(kx) \\ \psi_{III} &= 2B\cos(ka)e^{-\kappa(x-a)} \end{aligned}\right\} k\tan(ka) = \kappa$$

The constant B can be found from the normalization condition:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x = 1 \tag{21}$$

The combination of

$$k^2 + \kappa^2 = \frac{2m|U_0|}{\hbar} \tag{22}$$

and

$$k\cot(ka) = -\kappa \tag{23}$$

or

$$k\tan(ka) = \kappa \tag{24}$$

are only satisfied when k and κ and therefore E take on specific values, giving quantized energy levels.