The Hydrogen Atom

$$V(r) = \frac{-ke^2}{r}$$ - Coulomb potential

use spherical coordinate system

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi - \frac{ke^2}{r} \Psi = E \Psi$$

in spherical coordinates

$$\nabla^2 \Psi = \frac{1}{r^2 \sin\theta} \left[\sin\theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{\sin\theta} \frac{\partial^2 \Psi}{\partial \phi} \right]$$

The ground state will have the least 'wiggles' and be spherically symmetric -

$$\Psi(r, \theta, \phi) = f(r)$$

Hydrogen Atom ground State solution

we think lowest energy state will be spherically symmetric

so Schrodinger equation reduces to

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) - \frac{ke^2}{r}\psi = E\psi$$

guess that $\psi(r) \sim C e^{-r/\delta}$

$$\frac{d\psi}{dr} = -\frac{1}{\delta}\psi$$

$$\frac{d^2\psi}{dr^2} = \frac{1}{\delta^2}\psi$$

$$\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{d}{dr}\left(r^2 \times \frac{-1}{\delta}\psi\right) = -\frac{2r}{\delta}\psi + \frac{r^2}{\delta^2}\psi$$

Sub into Schr. eq.

$$-\frac{\hbar^2}{2mr^2}\left[-\frac{2r}{\delta} + \frac{r^2}{\delta^2}\right] - \frac{ke^2}{r} = E$$

r dependence must vanish on left, since E is a constant

$$\frac{\hbar^2}{m\delta} = ke^2 \qquad \delta = \frac{\hbar^2}{mke^2} \text{ - Bohr Radius}$$

leaving $$E = -\frac{\hbar^2}{2m\delta^2} = \frac{m(ke^2)^2}{2\hbar^2} = -\frac{\alpha^2 mc^2}{2} = -13.6\ eV$$

$$\alpha = \frac{2\pi ke^2}{hc} = \frac{ke^2}{\hbar c} \approx \frac{1}{137}$$